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THE EFFECT OF SURFACE-EVAPORATION KINETICS ON SUBLIMATION NEAR THE LEADING EDGE*

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Abstract—The behavior of the laminar binary boundary layer with blowing is coupled with the kinetic-evaporation-rate law of the surface material. It is found that the solution of this combined problem exhibits an asymptotic behavior at large Reynolds numbers which is identified with the usual near-equilibrium solution in which the vapor near the wall is very nearly in thermodynamic phase-change equilibrium. However, the near-equilibrium solution is invalid in a region near the origin of the boundary layer, which is characterized by a length formed with physical parameters describing the basic flow and the surface properties. In this region, which is treated here approximately, the blowing parameter decreases to zero, the wall temperature increases, and the sublimation rate tends to an upper limit as the origin of the boundary layer is approached.

The principal result of the analysis is the estimation of the length of the region of transition to the near-equilibrium solution. This length is an independent characteristic of the problem and does not scale as do the boundary-layer properties (that is, with the Reynolds number). This implies that while the transitional length on typical re-entry vehicles is small and probably negligible unless the leading edge is very sharp, practical wind-tunnel test models can easily be affected over most of their chord.

NOMENCLATURE

B', B, blowing parameters;
C, concentrations [mols/mol of mixture];
C_h, Stanton number, $q/\rho_e u_e(h_{aw} - h_w)$;
c_p, specific heat;
H, dimensionless parameter, $\lambda M_s/RT_{aw_0}$;
h, enthalpy;
k, dimensionless parameter $= 1 - T_T/T_{aw_0}$;
M_i, molecular weight of component i ($i = s$ for sublimating species, $i = A$ for free-stream air);
$\mathcal{M}$, free-stream Mach number;
m_s, mass rate of sublimation;
P, static pressure;
Pr, Prandtl number,
p, surface-vaporization (thermodynamic) constant, see equation (12);
q, heat-transfer rate;
R, universal gas constant;
T, temperature;
T_{aw}, adiabatic wall-recovery temperature [T_w for $k(\partial T/\partial y)_w = 0$];
X, dimensionless parameter, see equation (27);
x, streamwise distance (origin at stagnation point);
α, constant defining the boundary-layer transfer properties [equation (2)];
β, constant defining the recovery-temperature variation with blowing [equation (5)];
Γ, the factor $\bar{\gamma} - 1/\bar{\gamma}$;
γ, isentropic exponent of the gas;
δ, leading-edge effect scale, equation (41) [units of length];
ϵ, vaporization coefficient [equation (11)];
ζ, heat-transfer parameter, defined by equation (44);
λ, effective heat of sublimation;
μ, coefficient of viscosity;
ξ, heat-transfer parameter, for reference conditions without blowing [equation (20)] [units of pressure];

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ρ, density of the gas;
σ, parameter defined in equation (30);
τ, dimensionless parameter [see equation (18)].

Subscripts

e, free-stream conditions;
0, reference conditions without blowing;
T, stagnation conditions;
w, wall conditions;
∞, conditions existing asymptotically far downstream.

Subscript s pertains to sublimating species.

A bar over a symbol indicates average over binary mixture in the boundary layer.

I. INTRODUCTION

THE COMPLETE solution of a flow field over a subliming surface represents an equilibrium among the rate of heat transfer to the wall, the rate of phase change of the surface material, and the rate of diffusion of the vapor evolved at the wall through the boundary layer. These conditions determine the surface temperature and the blowing rate.

The problem is usually treated as follows: The binary boundary-layer equations are solved with the wall temperature and the blowing rate treated as independent boundary conditions. The solution yields the temperature gradient, that is, the heat-transfer rate, as a parametric function of the wall temperature and the blowing rate. Since the blowing rate and the heat transfer are related by the effective heat of sublimation of the material, one more relation is needed to fix uniquely the two free parameters (wall temperature and blowing rate). At this point we make the approximation that the phase change occurs at thermodynamic equilibrium; that is, the wall temperature is the phase-equilibrium temperature at the existing partial pressure of the vapor near the wall (which is known from the solution of the binary boundary-layer equations). This relation suffices to complete the formulation.

The assumption of thermodynamic phase equilibrium at the surface is conceptually incorrect because at equilibrium the net mass transfer between phases is zero. Therefore, a more complete kinetic relation among the temperature, concentration, and rate of phase change is actually needed. This has been discussed in a number of papers, [1–5] but no solution including such a kinetic surface-evaporation condition has been obtained, nor have the implications of this phenomenon been fully explored.

The near-equilibrium solution is valid at sufficiently high Reynolds numbers, which can be illustrated by saying that when the Reynolds number is high, the impedance of the boundary layer to diffusion of vapor is very much higher than the impedance to surface phase change, which is in series with it [4]. The latter is then negligible, and the coupling between the flow and the surface-evaporation phenomenon is diffusion limited. This statement leads to the question, What exactly are the lower limits of validity of this approximation?

If the surface-evaporation-rate law is considered, the results of the near-equilibrium approximation must be interpreted as saying that the *difference* between the actual wall temperature and the thermo-dynamic-equilibrium temperature of the vapor is small everywhere. The actual wall temperature is determined by the rate equation so as to supply the mass flux from the solid to the vapor phase. The equilibrium temperature is determined by the partial pressure which must exist near the wall to drive the mass flux across the boundary layer by pressure diffusion. Smallness is measured in comparison with the temperature difference driving the heat transfer; that is, the difference between adiabatic recovery and wall temperature.

The well-known result of near-equilibrium solutions for self-similar flows, for instance, is that the surface (equilibrium) temperature is constant, while the sublimation rate varies as the inverse square root of the Reynolds number. We may consider the Reynolds number as a unique measure of the distance from the origin of the boundary layer if the free stream is fixed and the wall temperature is constant. But if the mass flux increases towards the origin of the boundary layer, then the wall temperature must increase according to the sublimation-rate law. The wall temperature cannot be constant, and therefore the Reynolds number cannot be a unique measure of distance. Also, the near-equilibrium

approximation must break down at some distance towards the origin.

The properties of the boundary-layer scale in terms of two parameters, the Reynolds number and the wall temperature (or more correctly, some ratio of free-stream temperature to wall temperature). Having recognized that the wall temperature is coupled to the problem through an independent rate equation, one concludes that the introduction of the kinetics of surface phase change brings in a new scale parameter. This scale has the form of a distance from the origin of the boundary layer for fixed free-stream conditions and given thermodynamic phase-change constants.

The condition for validity of the near-equilibrium approximation can now be stated as follows: The approximation is valid when the Reynolds number is much higher than the Reynolds number formed with the leading-edge scale described above. Moreover, the illustrative argument based on the relative magnitude of the impedance to mass transfer is seen to be incomplete; it does not consider the nonlinear coupling between the "impedance" and the driving potential. For instance, it implies that when the Reynolds number is very low the boundary-layer transfer impedance is low and the coupling between flow and sublimation is dominated by the surface impedance (rate limited). This is not true if the Reynolds number is low by virtue of low pressure at a given distance from the origin. It is only true if the Reynolds number is low by virtue of small distances from the leading edge.

The purpose of this study is to investigate the character of the rate-limited sublimation problem and, in particular, to determine the scale of this region. It is proper to make two remarks at this point. First, the analysis is based entirely on continuum boundary-layer concepts. This is equivalent to saying that the scale of the rate-limited region must be larger than some minimum distance required for validity of continuum boundary-layer concepts for the analysis to be meaningful. This situation turns out to be practically possible. Second, in regard to practical flows over somewhat blunted bodies, the statement "origin of the boundary layer" must be interpreted as meaning a virtual origin from which the boundary layer would start to attain a thickness and profile it has at the point under consideration. This implies that the distance to this virtual origin must be larger than the radius at the blunted nose.

The current trend towards finer re-entry shapes and the concern with problems such as the interaction between ablative mass addition and hypersonic viscous induced pressure [6]—typically a leading-edge phenomenon—tends to bring the transitional sublimation regime into the realm of practical problems.

Finally, the present solutions for the rate-limited sublimation region involve a number of approximations in the treatment of the boundary layer and should be interpreted mainly as a study of whether or not the problem is sufficiently significant to deserve a more rigorous and much more difficult analysis.

II. FORMULATION

The problem requires the simultaneous solution of both the boundary layer and surface phase-change rate equations. Phenomenologically, these fall into four groups describing, respectively, the energy and the mass-transfer properties of the boundary layer and the surface.

The first is a solution of the classical binary boundary-layer equations which we consider to be uncoupled from the mass-diffusion equation [7], implying a Lewis number approximately equal to one. We do not consider chemical reactions between the interdiffusing species. The solution is taken in the approximate form

$$\frac{C_h}{C_{h_0}} = 1 - \frac{\alpha}{Pr} B \tag{1}$$

$$\alpha = \alpha\left(\frac{M_s}{M_A}, \frac{dp}{dx}, \frac{dT_w}{dx}\right) \tag{2}$$

where B is a normalized blowing parameter

$$B = \frac{\dot{m}_s}{\rho_e u_e C_{h_0}}. \tag{3}$$

The subscript 0 indicates reference conditions (non-ablating surface), and the Stanton number C_h is defined on the basis of the adiabatic recovery enthalpy (temperature):

$$C_h = \frac{q}{\rho_e u_e (h_{aw} - h_w)} = \frac{q}{\rho_e u_e \bar{c}_p (T_{aw} - T_w)} \quad (4)$$

Equation (1) is the well-known linear approximation derived from similarity solutions which holds to values of B on the order of 0·3. The numerical value of the proportionality constant and its dependence on Prandtl number, mass ratio, and pressure gradient (within the similar-flows family) has been discussed exhaustively in the literature [1, 7–10]. In the context of the discussion which follows, it is to be considered as a formal approximate expression of the behavior of the Stanton number for *B sufficiently small*, with α an unspecified function of the indicated parameters, applicable also to non-similar flows. It will be seen *a posteriori* that B goes from 0 when $Re_x = 0$ to an asymptotic value B_∞ when $Re_x \to \infty$. In the initial region the wall-temperature gradient is very strong and equation (1) is only defendable as the first term of an expansion about $B = 0$. In the region of asymptotic approach to equilibrium gradients are small* and, provided $B_\infty < 0{\cdot}3$, equation (1) holds as a "local similarity" solution. It is difficult to imagine a situation in which essential errors in the trends exhibited in the transition region would result from the use of this approximation.

To the same degree of approximation the recovery factor for the binary boundary layer is expressed in terms of the recovery factor for the reference flow:

$$r = \frac{T_{aw} - T_w}{T_{Te} - T_w} = r_0 - \frac{\beta}{Pr} B \quad (5)$$

The numerical values of $\beta = \beta(Pr, \mathrm{d}p/\mathrm{d}x, \mathrm{d}T_w/\mathrm{d}x, M_A/M_s)$ are less well defined from available solutions even for self-similar flows. It will be seen later, however, that the proportionality constants α *and* β *appear in the solution only as a ratio.* Thus, if they both vary in the same way with M_s/M_A, Pr, $\mathrm{d}p/\mathrm{d}x$, and $\mathrm{d}T/\mathrm{d}x$ to a first approximation, the effect of this dependence is minimized.

* The external pressure gradient is an independent parameter. It is assumed to be such that use of the "local-similarity" concept can be justified in regard to it.

The thermal coupling between the flow field and the boundary is specified by

$$q = \dot{m}_s \lambda_s \quad (6)$$

where λ_s is the heat of sublimation of the wall material. This approximation neglects radiant and conductive heat losses. It also neglects the heat needed to bring the sublimating material from its initial temperature to sublimation temperature; both are fair approximations for low-temperature sublimators.

If we define

$$B' \equiv \frac{\dot{m}_s}{\rho_e u_e C_h} = \frac{(h_{aw} - h_w)}{\lambda_s} \quad (7)$$

and use the identity

$$\frac{B}{B'} = \frac{C_h}{C_{h_o}} = \frac{T_{aw_o} - T_w}{(T_{aw} - T_w)} \quad (8)$$

we obtain the second well-known relation of simple theories.

The solution of the boundary-layer diffusion equation for the transfer of the sublimated material away from the wall is [7]

$$-\frac{C_s - C_{sw}}{C_{sw}} = \frac{(h_T - h_w)}{(h_{Te} - h_w)}. \quad (9)$$

This form is strictly true only for the case where both Le and Pr are equal to one; in this case the diffusion equation and the equation for the distribution of total enthalpy in the boundary layer are identical and concentration and energy profiles are similar. The effect of Pr on the similarity of the profiles is minor [7, 10], weaker than its effect on C_h itself. This justifies the use of equation (8) without also setting $Pr = 1$ in equation (1) and equations derived from it.

Combining equation (9) with Fick's law [7], we obtain an expression for the wall concentration as a function of the rate of sublimation at the wall:

$$\dot{m}_s(1 - C_{sw}) = C_{sw} \frac{q}{h_{Te} - h_w} = C_{sw} \rho_e u_e \frac{h_{aw} - h_w}{h_{Te} - h_w} C_h. \quad (10)$$

A last equation couples the concentration of the sublimated species in the boundary layer to

the sublimation phenomenon itself. The net rate of exchange of surface material across the surface-potential barrier is

$$\dot{m}_s = \epsilon\sqrt{\left(\frac{M_s}{2\pi RT_w}\right)}(P_{s\,\mathrm{equ}} - P_{sw}) = \epsilon\frac{P\bar{M}}{\sqrt{(\pi R M_s T_w)}}(C_{s\,\mathrm{equ}} - C_{sw}) \quad (11)$$

where ϵ is an empirical "vaporization coefficient", $P_{s\,\mathrm{equ}}$ and $C_{s\,\mathrm{equ}}$ are the equilibrium partial pressure and concentration corresponding to the wall temperature, and P_{sw} and C_{sw} are the actual partial pressure and concentration of the subliming material immediately over the surface in the boundary layer. For a two-component mixture one has the following auxiliary relations between the concentrations and the molecular weights ($\bar{M}$ is the mean molecular weight of the mixture):

$$\left.\begin{aligned}\frac{P_s}{P_e} = \frac{\bar{M}}{M_s}C_s = \left\{1 + \left[\frac{1}{C_s} - 1\right]\frac{M_s}{M_A}\right\}^{-1}\\ \bar{M} = \left[\frac{C_s}{M_s} + \frac{1}{M_A} - \frac{C_s}{M_A}\right]^{-1}\\ C_A + C_s = 1\end{aligned}\right\} \quad (12)$$

An important characteristic of the phenomenon is that there is a maximum rate of escape of surface atoms, which occurs when the concentration of the material in the surrounding gas is zero and which depends only on the wall temperature [2]. Several analytical expressions for the value of $P_{s\,\mathrm{equ}}$ that determines this maximum can be written down, depending on the subtlety of the microscopic model. The simplest one, corresponding to the integral of the Clausius–Clapeyron equation, is

$$P_{s\,\mathrm{equ}} = p\exp\left(-\frac{M_s\lambda_s}{RT_w}\right) \quad (13)$$

where p is a constant. This yields for the maximum (forward) vaporization rate the expression

$$(\dot{m}_s)_{\max} = \epsilon p\sqrt{\left(\frac{M_s}{2\pi RT_w}\right)}\exp\left(-\frac{M_s\lambda_s}{RT_w}\right). \quad (14)$$

The set of equations (1), (3), (9) and (13), together with an auxiliary caloric equation of state defining $\bar{c}_p$

$$\bar{c}_p = \frac{h_{aw} - h_w}{T_{aw} - T_w} \quad (15)$$

complete the general definition of the problem. We shall assume in this anslysis that $\bar{c}_p$ can be treated as an *a priori* (or iteratively) determined constant, not dependent on the concentrations [11].

The following dimensionless parameters, which depend only on the properties of the surface and/or the free-stream conditions, are now defined:

$$H = \frac{\lambda_s M_s}{RT_{aw_0}} \quad (16)$$

$$\Gamma = \frac{R}{M_A\bar{c}_p} = 1 - \frac{\bar{c}_v}{\bar{c}_p} = \frac{\bar{\gamma} - 1}{\bar{\gamma}} \quad (17)$$

$$\tau = -\frac{T_T - T_e}{T_{aw_0}}\frac{\beta}{Pr} \quad (18)$$

$$k = 1 - \frac{T_T}{T_{aw_0}} = \frac{\frac{\gamma - 1}{2}\mathscr{M}^2(r_0 - 1)}{1 + \frac{\gamma - 1}{2}\mathscr{M}^2 r_0} \quad (19)$$

In addition, define the function ξ (dimensions of pressure)

$$\xi = \frac{1}{\epsilon}\sqrt{\left(2\pi\frac{R}{M_s}T_{aw_0}\right)}\rho_e u_e C_{h_0}\sqrt{(Re_x)}. \quad (20)$$

The function ξ represents the product $C_{h_0}\sqrt{(Re_x)}$ which, at least for self-similar flows, is a constant. Otherwise this product varies with x through the streamwise pressure and the wall-temperature gradients. While questions must be raised as to its behavior near the singular point $x = 0$, which are mentioned again later, at large Reynolds numbers it is certainly a finite, slowly varying quantity.

Eliminating $\dot{m}_s$ between equations (3) and (11) and rearranging, we write

$$\frac{\rho_e u_e C_{h_0}}{\epsilon}\sqrt{\left(\frac{2\pi RT_{aw_0}}{M_s}\right)}B = \sqrt{\left(\frac{T_{aw_0}}{T_w}\right)}[P_{s\,\mathrm{equ}} - P_{sw}] \quad (21)$$

By straight substitutions we transform it into an equation in only one unknown, which is B:

$$\frac{B\xi}{\sqrt{(Re_x)}} = \left[\frac{1-(a/Pr)\,B}{1+B[\tau-(M_A/M_s)\,\Gamma H-(a/Pr)]-(a\tau/Pr)\,B^2}\right]^{1/2} \left[pe^{-H}\exp\left(H\,\frac{B[\tau-(M_A/M_s)\,\Gamma H]-(a\tau/Pr)\,B^2}{1+B[\tau-(M_A/M_s)\,\Gamma H-(a/Pr)]-(a\tau/Pr)\,B^2}\right)\right.$$
$$\left. -P_e\left(\frac{k+B[\tau-(M_A/M_s)\,\Gamma H-k\,(a/Pr)]-(a\tau/Pr)\,B^2}{k+B[\tau-(M_A/M_s)\,\Gamma H-k\,(a/Pr)]-(a\tau/Pr)\,B^2+\Gamma H[(a/Pr)\,B-1]}\right)\right] \quad (22)$$

III. ASYMPTOTIC BEHAVIOR DOWNSTREAM

For large values of Re the left-hand side of equation (22) vanishes and the equation yields a non-zero* value $B = B_\infty$ given by the solution of the factor in brackets on the right-hand side.

Note that this corresponds exactly to stating

$$[P_{s\,\text{equ}} - P_{sw}] \simeq 0$$

The solution is identified with the "equilibrium solution" in which the partial pressure of the vapor, and consequently also its concentration and temperature, are values corresponding to thermodynamic phase-change equilibrium.

In the present formulation the solution is given in terms of the constant p in the analytical expression for $P_{s\,\text{equ}}$ (instead of specifying $T_w = T_{s\,\text{equ}}$ separately from thermodynamic tables). A convenient graphical procedure is obtained defining

$$X_\infty = \frac{T_w}{T_{aw_0}} - 1 = \frac{B_\infty[\tau-\Gamma H\,(M_A/M_s)]-B_\infty^2\,(a/Pr)\,\tau}{1-(a/Pr)\,B_\infty} \quad (23)$$

in terms of which the asymptotic solution takes the form

$$\ln\frac{P_e}{p} = -\frac{H}{X_\infty+1} + \ln\left[\frac{X_\infty+k-\Gamma H}{X_\infty+k}\right] \quad (24)$$

This is plotted in Fig. 1 for a particular value of Γ. For any external static pressure P_e and parameters describing the properties of the sublimating material (p and H), one obtains a unique value for the wall-temperature function X_∞. With this value of X_∞ and further parameters pertaining to the reference flow and the sublimating material ($a/Pr\tau$), Fig. 2, which is a plot of equation (23), yields the value of the blowing parameter B_∞.

* The singularity $B = 0$ is uninteresting. It corresponds simply to flow without sublimation.

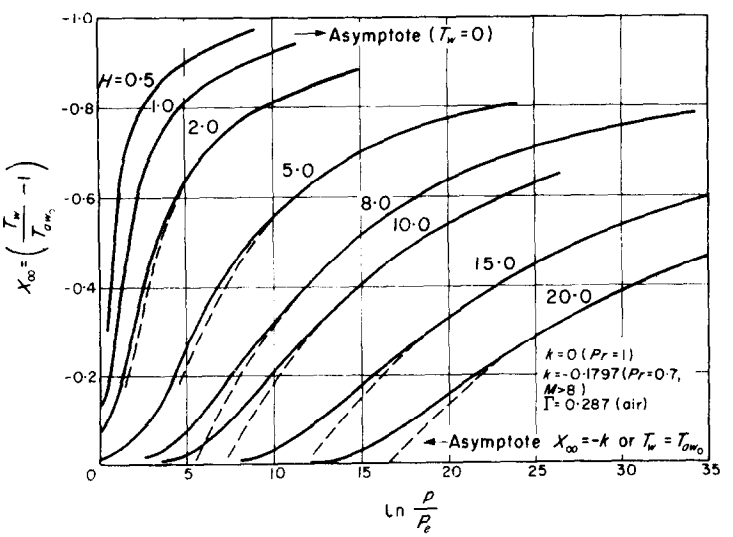

FIG. 1. Wall-temperature function.

Figure 3 is an auxiliary figure giving the variation of $\alpha/Pr\tau$. This can be written as

$$\frac{\alpha}{Pr\tau} = \frac{\alpha/\beta}{(1-k)\,[1-(T_e/T_T)]}. \tag{25}$$

The two proportionality constants α and β [see equations (2) and (5)] appear as a ratio. Therefore, their dependence on the principal variables

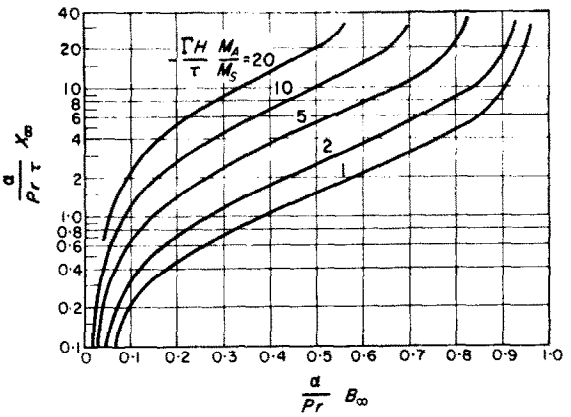

FIG. 2. Asymptotic blowing parameter vs. wall-temperature function.

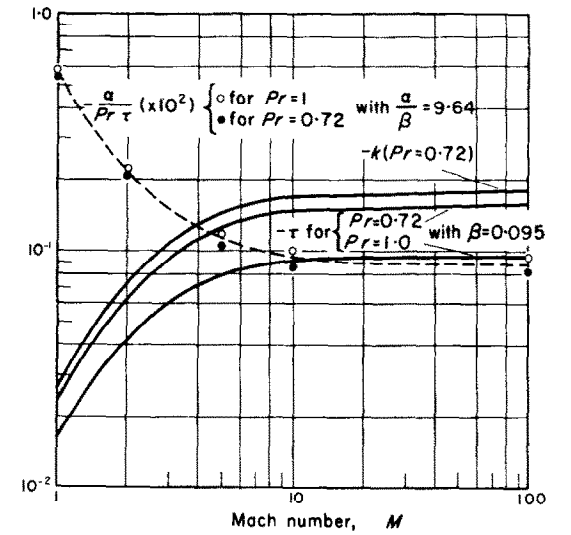

FIG. 3. Variation of some Mach-number-dependent parameters.

of the binary boundary-layer problem, which are M_s/M_A, Pr, pressure and wall-temperature gradient, is minimized. If one makes the hypothesis that both vary in the same fashion with these parameters, which seems probable, then the combination $\alpha/Pr\tau$ is to a very good approximation a function only of the free stream. Using the flat-plate solution given in reference 9, the numerical value of the ratio α/β is 9·64.

IV. BEHAVIOR NEAR THE STAGNATION POINT

Equation (22) has the form

$$\frac{\xi}{\sqrt{(Re_x)}} = \frac{f(B)}{B}. \tag{26}$$

The parameter ξ appearing on the left-hand side of equation (26) represents the product $C_{h_0}\sqrt{(Re_x)}$. The assumption is made that this product is a constant, as it is for self-similar boundary-layer solutions. Now, since $q = \lambda \dot{m}_s$ is bounded according to the surface-evaporation equation, the assumption implies that the wall temperature tends to the recovery temperature towards the leading edge ($Re_x \to 0$). It follows that the wall temperature is variable, and $C_{h_0}\sqrt{(Re_x)}$ = constant can only be good in the sense of "local similarity".

It is worth noting that the above implies two statements of unequal importance to the present analysis. The more important one concerns the *behavior* of C_{h_0}; that is, that C_{h_0} grows without bounds towards the leading edge. This leads to the result that $B \sim \dot{m}_s/C_{h_0}$ tends to zero there and, thus, that B varies between zero and B_∞ over the entire region of interest. The second and less important one concerns the numerical accuracy of the assumption $C_{h_0}\sqrt{(Re_x)}$ = constant. It is undoubtedly poor very near the origin where the wall-temperature gradients are large, but it is probably satisfactory in the region of asymptotic approach to the near-equilibrium solution downstream of the leading edge.

It is not possible to discuss conclusively the difficult problem of the singularity at the leading edge. It must be accepted on the basis of heuristic arguments and the analogy with the behavior of ordinary boundary layers at the leading edge, which involves similar difficulties. Physically, the behavior outlined in what preceded is quite reasonable. Moving upstream towards the leading edge, the heat flux to the wall increases, and the wall temperature must rise to permit an increased rate of sublimation.

Returning to equation (26), the right-hand side is expressible in the form of a series in the interval $0 < B < B_\infty$. If the asymptotic blowing parameter B_∞ is reasonably small, which is already implied by the use of the linear-blockage equation, equation (1), a limited number of terms of the series will represent the behavior of the function throughout the range.

The series is

$$\frac{\xi/f(0)}{\sqrt{(Re_x)}} = \frac{1}{B} + \frac{f'(0)}{f(0)} + \frac{1}{2}B\frac{f''(0)}{f(0)} + \dots \quad (27)$$

with the following expressions for the function and its derivative evaluated at the origin:

$$f(0) = pe^{-H} - \frac{kP_e}{k - \Gamma H} \quad (28)$$

$$\frac{f'(0)}{f(0)} = \left(\tau - \Gamma H \frac{M_A}{M_s}\right)\left[\frac{3}{2} + \sigma \frac{P_e}{\bar{p}} \frac{e^H}{\Gamma H}\right] \quad (29)$$

The parameter σ has the form

$$\sigma = \frac{(\Gamma H)^2 pe^{-H} + k(k - \Gamma H)\Gamma H pe^{-H}}{(\Gamma H)^2 pe^{-H} + k(k - 2\Gamma H)pe^{-H} + kP_e(\Gamma H - k)} \quad (30)$$

and it was defined so as to become one when $k = 0$ (Prandtl number unity).

The first two terms of the series expansion, tend to the asymptotic limit of $f(0)/f'(0)$. As the Reynolds number increases, this limit is not correct because the ratio $f(0)/f'(0)$ does not equal B_∞ unless B_∞ tends to zero. We know B_∞ independently from equations (23) and (24), or Figs. 1 and 2.

The argument suggests that if one replaces in the two-term expansion

$$-\frac{f'(0)}{f(0)} \text{ by } \frac{1}{B_\infty} \quad (31)$$

one obtains a good approximation to the behavior of B for all Re_x, provided B_∞ is sufficiently small to justify the linear expression for the blockage factor, equation (1).

Accordingly, an approximate equation for B is as follows:

$$\frac{B}{B_\infty} = \frac{\sqrt{(Re_x)}}{\sqrt{(Re_x)} + \sqrt{\Delta}} = \frac{\sqrt{(x/\delta)}}{\sqrt{(x/\delta)} + 1} \quad (32)$$

where Δ or δ are scale factors defined by:

$$\Delta = \left[\frac{\xi B_\infty}{f(0)}\right]^2 = \delta \frac{\rho_e u_e}{\mu_e} \quad (33)$$

Δ is dimensionless. δ has dimensions of length. It can be given explicitly as a function of reference-flow sublimating-material parameters by*

$$\delta = B_\infty^2 \left(pe^{-H} - \frac{kP_e}{k - \Gamma H}\right)^{-2} \frac{2\tau}{\epsilon^2} \frac{R}{M_s} \mu_e u_e \rho_e T_{aw_0} [C_{h_0}\sqrt{Re_x})]^2. \quad (34)$$

Equation (32) is compared to the numerical solution of the full equation (22) on Fig. 4 for a particular example.

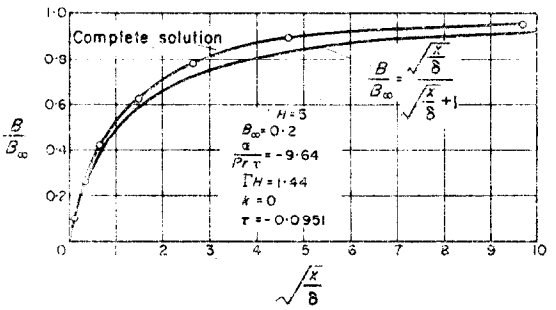

FIG. 4. Comparison of full solution with the approximate modified first-term expansion.

By straightforward substitutions among the basic equations (1), (3), (9), and (13) all physical parameters of the problem can be written out in terms of B. If these are then expanded in series and B as given by equation (32) is substituted in the first term of the expansion, one obtains explicit expressions of their variation near the origin of the boundary layer. As an example, the equation for the blowing rate is:

$$\dot{m}_s = \frac{\epsilon f(0)}{\sqrt{\left(2\pi \frac{R}{M_s} T_{aw_0}\right)}} \frac{1}{\sqrt{\left(\frac{x}{\delta}\right) + 1}}. \quad (35)$$

This equation shows that $\dot{m}_s$ starts at the leading edge from its maximum value (dictated by kinetic considerations) and decreases downstream to its asymptotic behavior, which is

* Note that $C_{h_0}\sqrt{(Re_x)}$ is a constant with x.

proportional to the inverse square root of the streamwise distance.

Similar explicit expressions for the variation of the other parameters are given in more detail in reference 13. The concentration and the partial pressure of the sublimating species have at the origin values which depend only on k (the Prandtl number) and which are zero for the particular case of $Pr = 1$. They increase downstream. The wall temperature at the leading edge is the adiabatic recovery temperature for no ablation, that is, if $Pr = 1$, it is the free-stream stagnation temperature. It decreases downstream to its asymptotic value.

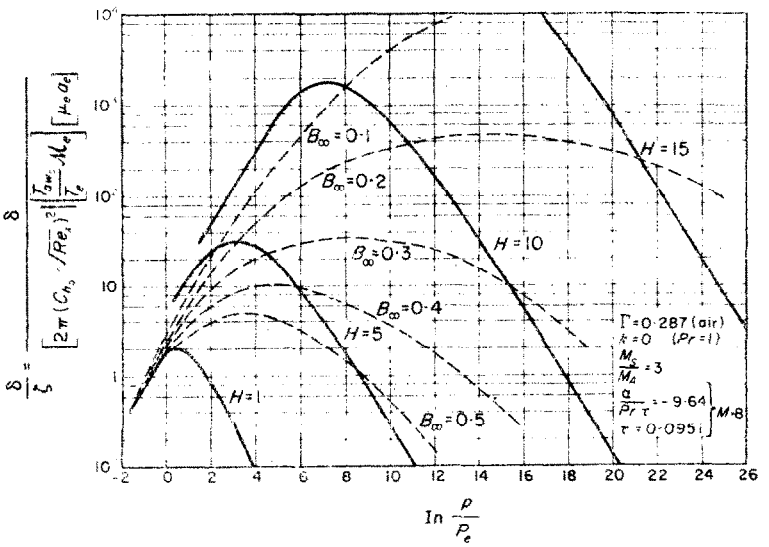

FIG. 5. Typical variation of the rate-controlled flow length, showing lines of B_∞ = constant.

It is of interest to consider further the parameters δ, which determines the scale of the adjustment to the asymptotic solution. From its definition and using the equation of state, one obtains

$$\frac{\delta}{\xi} \equiv \frac{\delta[p\epsilon^2\,(M_s/M_A)]}{[2\pi\{C_{h_0}\sqrt{(Re_x)}\}^2]\,[(T_{aw_0}/T_e)\mathscr{M}_e]\,[\mu_e a_e]} = \frac{p}{P_e}\left[\frac{\sqrt{(\Delta P_e)}}{\xi}\right]^2 = \frac{P_e}{p}\,B_\infty^2 \exp(2H) \quad (36)$$

This is plotted in Fig. 5. Note that the three bracketed terms of the denominator on the left-hand side depend only on the basic flow field, the Mach number, and the free-stream static temperature, respectively. The bracketed term in the numerator depends only on the properties of the sublimating material. The entire factor on the left-hand side is independent of pressure. Consequently, Fig. 5 shows that during the initial re-entry from space ($p/P_e = \infty$) during which the Mach number is roughly constant (therefore H = constant and ζ = constant), δ increases to a maximum at some altitude fixed by the value of the parameter p and then decreases.

VI. DISCUSSION

In order to show the magnitudes involved, consider the following estimates: Let the model be a slender cone for which [9]

$$C_{h_0}\sqrt{(Re_x)} = 0{\cdot}512.$$

Assume μ_e and a_e constant at their value in the tropopause ($\mu_e a_e = 4 \times 10^{-4}$ lb/ft). It follows that

$$[2\pi\{C_{h_0}\sqrt{(Re_x)}\}^2]\left[\frac{T_{aw_0}}{T_e}\mathscr{M}_e\right][\mu_e a_e] \simeq 1{\cdot}3\,\mathscr{M}^3 \times 10^{-4}\ \text{lb/ft} \quad (37)$$

The constant p can be obtained from vapor-pressure data. Reference 12 (pp. 1751–1755) gives a table for selected organic and inorganic

substances from which it is evident that a representative value is $p = 10^9$ lb/ft².* (The value p varies around this average by one order of magnitude for almost all the substances listed). The vaporization coefficient ϵ is poorly known. For solid sublimators it is doubtful that it should exceed 0·1 [2], and it can be less than that by two orders of magnitude. Finally, the ratio M_s/M_A can be taken as unity for the purpose of this estimate.

It follows that, quite generally,

$$10^{-11} < \frac{\zeta}{\mathscr{M}^3} < 10^{-9} \text{ ft.} \qquad (38)$$

The magnitude of δ is seen to depend very strongly on the value of H. Since it increases rapidly with H, let us illustrate the problem for a large value of H: Data typical of graphite ($\lambda = 25\,000$ Btu/lb, $M_s = 12$) at a flight Mach number of 15 yield approximately

$$H = \frac{\lambda M_s}{RT_{aw_0}} \simeq \frac{\lambda M_s}{RT_e}\left(1 + \frac{\gamma - 1}{2}\mathscr{M}^2\right)^{-1} \simeq 16{\cdot}7.$$

The value of $p = 10^9$ lb/ft² used in the preceding estimates represents graphite quite well—well enough considering the uncertainty in the evaporation coefficient ϵ. Since δ increases with pressure, let us consider relatively low altitudes consistent with the assumed Mach number and high-speed re-entries, say 30 000 ft. Consider a 20-deg cone for which the surface pressure is approximately 300 P_∞. It follows that

$$10^{-3} < \delta < 10^{-1} \text{ in.}$$

Had an altitude of 25 000 ft been assumed, δ would be larger by an order of magnitude. Had data typical of teflon been used ($\lambda \sim 1000$ Btu/lb, $M_s = 100$), H would have been in the order of 5, and δ would be entirely negligible.

It is interesting to note that the maximum δ for a given H occurs at values of p/P_e which correspond to P_e considerably higher than 1 atm. While pressures above atmospheric can occur, because p_e and $\mathscr{M}_e$ are values outside the boundary layer behind the leading shock, they would not normally be as high as indicated for δ max.

* In the notation of the reference, $p = 10^b$ where b is tabulated.

The rate-controlled region can be taken to extend over a distance from the origin equal to 100 δ; that is, for points on the surface lying beyond this limit the error incurred by using the equilibrium solution for B, $\dot{m}_s$, T_w, etc., is less than 10 per cent. Figure 6 shows the behavior of the rate of sublimation indicated by both the equilibrium approximation and the full solution and illustrates the present argument.

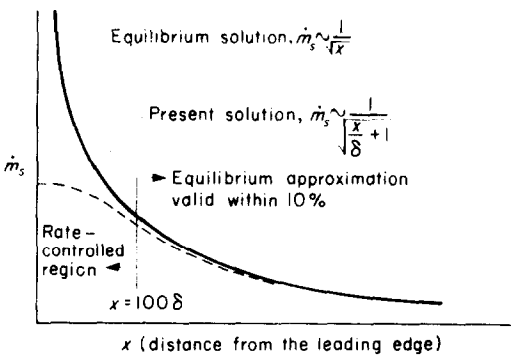

FIG. 6. Schematic variation of the sublimation rate.

The preceding estimates of the scale of the leading-edge effect indicate clearly that, *in comparison with reasonable man-made re-entry-vehicle sizes, the region of transitional sublimation is very small,* indeed normally negligible. However, δ *is an independent parameter which does not scale with the body geometry.* The sublimation of *small models in wind tunnels or meteorites in the atmosphere may be totally dominated by transitional effects.*

This scaling problem can best be demonstrated by using a concrete example. Consider the wind-tunnel experiments reported in reference 3 using camphor in a Mach 5 wind tunnel. For camphor the materials data at actual test conditions can be given quite accurately, except for ϵ:

$$\lambda = 330 \text{ J/g}, \quad M_s = 152, \quad p = 1{\cdot}74 \times 10^9 \text{ lb/ft}^2.$$

Estimate ϵ at 0·1 as before.

With the tunnel recovery temperature at 350°K and the static pressure of 10 mm Hg, one finds $H = 17{\cdot}2$ and $p/P_e = 6{\cdot}3 \times 10^7$:

$$\delta \simeq 2{\cdot}6 \times 10^{-2} \text{ in.}$$

Considering that the transitional sublimation region extends to a distance of the order 100 δ, one concludes that the *entire wind-tunnel model* is affected by transitional effects under these test conditions.

One other aspect of the aerodynamics of subliming bodies can be affected by the present results in an important manner. It is the problem of surface recession and terminal shape. Briefly, the shape of a slender body $y_{(x,\,t)}$ at any instant t is given by the solution (with proper boundary conditions) of

$$\frac{\partial y_{(x,t)}}{\partial t} \sim \dot{m}_{s(x,t)} = \frac{K}{(\sqrt{X^1}) + \sqrt{\delta}} = \frac{K}{\sqrt{(x - \phi_{(t)}) + \sqrt{\delta}}}$$

where K is a constant, X^1 the distance to the point (x, y) from the *leading edge*, which is itself receding relative to fixed coordinates at an unknown rate $\phi_{(t)}$.

The nature of this problem is such that

$$\lim_{\delta \to 0} y_{(x,t,\delta)} \neq y_{(x,t,\delta=0)}$$

and the analytical solution for the terminal shape of pointed bodies using the near-equilibrium form of the local sublimation rate is not correct.*

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* A study of this problem is in progress at The RAND Corporation.

Résumé—Le comportement de la couche limite laminaire binaire avec soufflage est couplé avec la loi de la cinétique d'évaporation du matériau de la surface. On trouve que la solution de ce problème combiné présente un comportement asymptotique à des nombres de Reynolds élevés qui s'identifie avec la solution habituelle près de l'équilibre dans laquelle la vapeur près de la paroi est très voisine de l'équilibre thermodynamique du changement de phase. Cependant la solution au voisinage de l'équilibre n'est plus valable dans une région près de l'origine de la couche limite, qui est caractérisée par une longueur formée avec des paramètres physiques décrivant l'écoulement de base et les propriétés de la surface. Dans cette région, qui est traitée ici approximativement, le paramètre de soufflage décroît vers zéro, la température pariétale augmente, et la vitesse de sublimation tend vers une limite supérieure lorsqu'on approche de l'origine de la couche limite.

Le résultat principal de l'analyse est l'estimation de la longueur de la région de transition vers la solution au voisinage de l'équilibre. Cette longueur est une caractéristique indépendante du problème et ne varie pas comme les propriétés de la couche limite (c'est-à-dire, avec le nombre de Reynolds). Ceci implique que tandis que la longueur de transition sur des véhicules typiques de rentrée est faible et probablement négligeable à moins que le bord d'attaque soit très aigu, des modèles pratiques d'essais en soufflerie peuvent facilement être affectés sur la plus grande partie de leur corde.

Zusammenfassung—Das Verhalten der laminaren, binären Grenzschicht mit Ausblasen wird mit dem Gesetz der kinetischen Verdampfungsgeschwindigkeit des Oberflächenmaterials gekoppelt. Es ergibt

sich, dass die Lösung dieses kombinierten Problems bei grossen Reynoldszahlen ein asymetrisches Verhalten aufweist, das der gebräuchlichen Lösung für nahezu erreichtes Gleichgewicht, wo der wandnahe Dampf nahe am thermodynamischen Gleichgewichtszustand des Phasenwechsels liegt, gleichgesetzt wird. Die Lösung für das nahezu erreichte Gleichgewicht ist jedoch in einem Bereich nahe dem Ursprung der Grenzschicht ungültig; dabei wird der Ursprung von einer Länge gekennzeichnet, die aus physikalischen Parametern, die die Grundströmung und die Oberflächeneigenschaften beschreiben, gebildet wird. In diesem hier näherungsweise behandelten Bereich geht der Ausblasparameter nach Null, die Wandtemperatur nimmt zu und die Sublimationsgeschwindigkeit tendiert nach einem oberen Grenzwert, wenn der Ursprung der Grenzschicht erreicht wird. Das Hauptergebnis der Analyse liegt in der Längenabschätzung für den Übergangsbereich zur Lösung für nahezues Gleichgewicht. Diese Länge ist eine unabhängige Kenngrösse des Problems und ändert sich nicht, wie es bei den Eigenschaften der Grenzschicht (d.h. mit der Reynoldszahl) der Fall ist. Dies schliesst ein, dass die Modelle für praktische Versuche im Windkanal leicht fast über ihre ganze Profiltiefe beeinflusst werden können, während die Übergangslänge an typischen Wiedereintrittsfahrzeugen klein ist und wahrscheinlich vernachlässigbar, wenn die Vorderkante nicht sehr scharf ist.

Аннотация—Поведение ламинарного бинарного пограничного слоя со вдувом взаимосвязано с законом, определяющим кинетику испарения материала поверхности. Найдено, что при больших значениях числа Рейнольдса решение этой взаимосвязанной задачи асимптотически совпадает с обычным решением для случая почти равновесного состояния системы, когда пар у стенки находится почти в термодинамическом фазовом равновесии. Однако, решение для случая околоравновесного состояния системы несправедливо для области возникновения пограничного слоя, которая характеризуется длиною, образованной из физических параметров основного потока и поверхности. В этой области, описанной в работе приближенно, параметр вдува уменьшается до нуля, температура стенки увеличивается, а скорость сублимации стремится к верхнему пределу по мере приближения к началу пограничного слоя.

Основным результатом настоящего анализа является оценка длины переходной области к околоравновесному решению. Эта длина является независимой характеристикой задачи и не изменяется в отличие от характеристик пограничного слоя (т.е. с критерием Рейнольдса). Отсюда следует, что тогда как длина перехода на обычных ракетах, возвращающихся в атмосферу, мала и возможно пренебрежимо мала, если только передняя кромка не очень острая, модели для продувки в аэродинамической трубе легко могут быть затронуты в большей части их поверхности.